Sheet 4

1. Deduce (afresh) the 'Weak Nullstellensatz':

- *if E is a finitely generated F-algebra where $E \supseteq F$ are fields, then $(E : F)$ is finite*

from the 'Noether Normalization Lemma'.

2. (i) Let F be an algebraically closed field. Show that $J(F[t_1, \ldots, t_k]) = 0$ (without quoting a general theorem about the Jacobson property of algebras!)

(ii) Show that if $R \subseteq S$ is an integral ring extension then $J(S) \cap R = J(R)$ (cf. sheet 3 q. 4(i)). Deduce that if, in addition, S is an integral domain, then $J(S) = 0$ if and only if $J(R) = 0$.

(iii) Now let F be an arbitrary field. Using the Noether Normalization Lemma, deduce that every finitely generated F-algebra is a Jacobson ring.

3. (i) Prove that $\mathbb{Q}$ is not finitely generated as a $\mathbb{Z}$-algebra.

(ii) Let F be a field, and suppose that F is finitely generated as a $\mathbb{Z}$-algebra. Prove that $\mathrm{char}(F) \neq 0$. (*Hint*: Suppose that F has characteristic 0. Consider the three rings $\mathbb{Z} \subseteq \mathbb{Q} \subseteq F$.)

(iii) Let S be a finitely generated $\mathbb{Z}$-algebra and M a maximal ideal of S. Prove that S/M is finite.

4. Let R be a subring of a field E and Y a multiplicatively closed subset of R with $1 \in Y$ and $0 \notin Y$. Let S be the integral closure of R in E. Prove that the integral closure of RY^{-1} in E is SY^{-1}.

An integral domain R is said to be *integrally closed* if R is its own integral closure in its field of fractions.

5. Let R be an integrally closed integral domain with field of fractions F, and $E \supseteq F$ an algebraic field extension. Show that for $a \in E$ the following are equivalent: (a) a is integral over R, (b) the (monic) minimal polynomial of a over F lies in $R[t]$. [*Hint*: consider a suitable splitting field.]

Does this necessarily hold if R is not integrally closed?

6. Let R be a Noetherian local integral domain, i.e. R has a unique maximal ideal $P \neq 0$. Assume (a) $h(P) = 1$ (see below) and (b) R is integrally closed. Prove that R *is a PID* as follows (or otherwise!)

(i) Let $0 \neq a \in P$. Show that for some $n \geq 1$ we have $P^{n-1} \nsubseteq aR$ and $P^n \subseteq aR$ (where $P^0 = R$).

Let $b \in P^{n-1} \setminus aR$ and put $y = a^{-1}b$. Show that if $yP \subseteq P$ then $y \in R$; deduce that in fact $yP \nsubseteq P$ (*Hint*: consider the action of y on the R-module $a^{-1}P$).

(ii) Now deduce (a) that $yP = R$ and hence (b) that P is a principal ideal.

(iii) Let $0 \neq I$ be a proper ideal of R. Prove that $I = P^n$ for some n. (*Hint*: show first that there is a maximal n for which $I \subseteq P^n$.)

Note: $h(P)$ is the maximal length n of a chain of prime ideals $P_0 < P_1 < \ldots < P_n = P$ (allowing $P_0 = 0$ iff R is an ID).

$\dim R$ is the supremum of $h(P)$ over all prime ideals P (or all maximal ideals, of course).

7. Let R be a ring (not necessarily Noetherian). Let P be a prime ideal of $S = R[t]$ with $t \in P$. Show that if $h(P/tS)$ is finite then $h(P) > h(P/tS)$. [*Hint*: show that if Q is a prime ideal of R then QS is prime in S].

Deduce that if $\dim R$ is finite then $\dim(S) > \dim R$.